DISTRIBUTION FUNCTION OF ISOTROPICALLY SCATTERED PARTICLES IN ANISOTROPIC STRUCTURES

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A study is made of the properties of solutions for a one-dimensional single-velocity transport equation for particles in a medium in which the probabilities of the elementary processes depend on the direction, and the scattering indicatrix (phase function) is spherical. It is shown that the linear parameter of the asymptotic exponential reduction in the global particle density when orientational inhomogeneity is allowed for can differ strongly from the value found by averaging the probabilities of the elementary processes over the directions. The role played by the structure of the orientational inhomogeneity is considered. The results are generalized to the case of anisotropy of the scattering.

To treat the transport of particles or radiation in a medium rigorously, one must solve the corresponding transport equation (see, for example, [1,2]). This equation has the simplest form when the scattering indicatrix (phase function) is spherical, a situation that arises in a number of cases. In solving transport equations, one generally assumes, implicitly or explicitly, that the properties of the medium do not depend on the direction in which a particle moves. But if we are concerned with anisotropic structures – for example, crystals of dispersed media with well-defined orientation – the behavior of a particle must obviously depend in some manner on the direction on which it is moving. However, it is impossible to say in advance how much the anisotropy affects the statistical characteristics of the transport process. Some light has been cast on this problem by Lindhard [3], who has considered orientational effects associated with the motion of charged particles in a crystal lattice. In this paper, we consider the one-dimensional problem of the distribution of isotropically scattered particles in an infinite medium in which the probabilities of scattering and absorption of particles are functions of the angle between the direction in which the particles move and the normal to a plane that is an isotropic source of particles. The results are also generalized to the case of anisotropic scattering.

If $f(x, \mu)$ is the particle distribution function, depending on the linear coordinate x and on μ, which is the cosine of the angle measured from the x axis, the transport equation, which describes the variation in x of the distribution function, can be written in the form

$$\mu \frac{\partial f(x,\mu)}{\partial x} + \Sigma(\mu) f(x,\mu) - \frac{1}{2}\int_{-1}^{1} \Sigma_s(\mu') f(x,\mu')\, d\mu' = \frac{1}{2}\delta(x) \tag{1}$$

Here, $\Sigma - \Sigma_S$ and Σ_S are the direction-dependent macroscopic absorption and scattering cross sections, respectively, and δ is the Dirac delta function. Note that the term "macroscopic cross section" is merely another name for the reciprocal mean free path in the given direction.

Equation (1) is solved by a method similar to those used to solve problems in which the cross sections do not depend on the angles [1,2]. Generally, we are only interested in the global particle density, i.e., in

$$\varphi(x) = \int_{-1}^{1} f(x,\mu)\, d\mu \tag{2}$$

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For the function calculated in accordance with (2), we obtain

$$\varphi(x) = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \left(1 - \frac{1}{2} \int\limits_{-1}^{1} \frac{\Sigma_s d\mu}{\Sigma + i\omega\mu}\right)^{-1} e^{i\omega x} \int\limits_{-1}^{1} \frac{d\mu}{\Sigma + i\omega\mu}\, d\omega \tag{3}$$

The expression (3) is the general solution of the problem for different dependences $\Sigma(\mu)$ and $\Sigma_S(\mu)$. It is a good idea to consider some special cases in which (3) reduces to perspicuous expressions that reveal the effect of the anisotropy.

We assume that $\Sigma(\mu)$ and $\Sigma_S(\mu)$ are even functions (the "forward" and "backward" directions are on an equal footing) and that $\Sigma_S\Sigma^{-1} = b = \text{const} < 1$ (the scattering and absorption events are due to the same centers).

Let

$$\Sigma = a \begin{cases} \Delta^2, & |\mu| < \Delta \\ \mu^2, & |\mu| > \Delta \end{cases} \tag{4}$$

Thus, we assume that the x axis coincides with the "least transparent" direction. The numerical value of the parameter Δ determines the range of variation in the values of the cross sections.

Substituting (4) into (3), we obtain

$$\begin{aligned} \varphi(x) = \frac{1}{4\pi i} \int\limits_{-\infty}^{\infty} \left(2 \ln \frac{\Delta a + i\omega}{\Delta a - i\omega} - \ln \frac{a + i\omega}{a - i\omega}\right) \Big\{1 - b(1 - \Delta) + \frac{b}{2} \\ \times \left[\frac{i\omega}{a} \ln \frac{a + i\omega}{a - i\omega} - \left(\frac{i\omega}{a} + \frac{\Delta^2 a}{i\omega}\right) \ln \frac{\Delta a + i\omega}{\Delta a - i\omega}\right]\Big\}^{-1} e^{i\omega x} \frac{d\omega}{\omega} \end{aligned} \tag{5}$$

The integrand in (5) has two simple poles on the imaginary axis, $\pm i\omega_0$, $\omega_0 < \Delta a$, which are found from a transcendental equation, and four branch points, $\pm ia$, $\pm i\Delta a$. The point $\omega = 0$ is not a pole. The integral (5) is made up of the contribution from the residue, J, at the pole $+i\omega_0$ and the integral, I, around a contour that circumvents the cut in the upper half-plane:

$$\varphi(x) = J + I \tag{6}$$

where

$$\begin{aligned} J = \frac{a}{2b\omega_0} \left\{2 \frac{1 - b(1 - \Delta)}{b} + \frac{\Delta^2 a^2 - \omega_0^2}{a\omega_0} \ln \frac{\Delta a - \omega_0}{\Delta a + \omega_0}\right\} \\ \times \left\{\frac{1 - b(1 - \Delta)}{b} + \frac{\Delta^2 a}{\omega_0} \ln \frac{\Delta a - \omega_0}{\Delta a + \omega_0} - \frac{\omega_0^2}{a^2 - \omega_0^2} + \frac{\Delta(a^2\Delta^2 + \omega_0^2)}{\Delta^2 a^2 - \omega_0^2}\right\}^{-1} e^{-\omega_0 |x|} \end{aligned} \tag{7}$$

$$\begin{aligned} I = \frac{1}{2} \int\limits_{1}^{\infty} \left\{1 - b(1 - \Delta) - \frac{b}{2}\left(y - \frac{\Delta^2}{y}\right)\left(\ln \frac{y - 1}{y + 1} - \ln \frac{y - \Delta}{y + \Delta}\right)\right\} \\ \times \left\langle \left\{1 - b(1 - \Delta) - \frac{b}{2}\left[y \ln \frac{y - 1}{y + 1} - \left(y + \frac{\Delta^2}{y}\right) \ln \frac{y - \Delta}{y + \Delta}\right]\right\}^2 + \frac{\pi^2 b^2}{4} \frac{\Delta^4}{y^2} \right\rangle^{-1} \\ \times e^{-ay|x|} \frac{dy}{y} + \frac{1}{2} \int\limits_{\Delta}^{1} \left\{2 - 2b(1 - \Delta) - \frac{b}{2}\left(y - \frac{\Delta^2}{y}\right) \ln \frac{1 - y}{1 + y}\right\} \\ \times \left\langle \left\{1 - b(1 - \Delta) - \frac{b}{2}\left[y \ln \frac{1 - y}{1 + y} - \left(y + \frac{\Delta^2}{y}\right) \ln \frac{y - \Delta}{y + \Delta}\right]\right\}^2 + \frac{\pi^2 b^2}{4}\left(y + \frac{\Delta^2}{y}\right)^2 \right\rangle^{-1} \\ \times e^{-ay|x|} \frac{dy}{y} \end{aligned} \tag{8}$$

The component (7) of the particle density is usually called the asymptotic component, since it determines the behavior of $\varphi(x)$ at large values of the argument [elementary estimates show that I, which is given by (8), decreases faster than J].

Thus, $\varphi(x)$ decreases exponentially with increasing distance from the emitting plane with characteristic linear parameter ω_0^{-1}, which, as in the isotropic case, may be called the diffusion length. An approximate solution of the transcendental equation for ω_0 is

$$\omega_0 \approx \Delta a \sqrt{3(1 - b)/b(4 - 3\Delta)} \tag{9}$$

This holds when $(1-b)/\Delta \ll 1$, i.e., when the absorption is weak. Thus, the diffusion length in the "least transparent" direction depends very strongly on the extent to which the scattering absorption cross sections are anisotropic. For example, if $\Delta = 0.7$, the diffusion length for the region in which (9) is applicable is about twice as large as for the isotropic case with the same value of a. If Δ is sufficiently small, the diffusion length varies as Δ^{-1}.

It is interesting to compare ω_0, the effective macroscopic cross section for the reduction of the particle density along the direction of the normal to the emitting plane in the asymptotic region, with the analogous mean value

$$\omega_1 = \frac{1}{2}\sqrt{\frac{3(1-b)}{b}}\int_{-1}^{1}\Sigma(\mu)\,d\mu$$

We find

$$\frac{\omega_0}{\omega_1} = \frac{3\Delta}{\sqrt{4-3\Delta}\,(1+3\Delta-\Delta^3)} \tag{10}$$

It can be seen from (10) that this ratio varies within wide limits when Δ varies. When $\Delta = 1$, the ratio is equal to unity, as it must; when $\Delta \ll 1$, the ratio is $\approx 3\Delta/2$. Thus, if Δ is small, i.e., the "transparency" is strongly anisotropic, the particle density decreases much more slowly than one would expect from averaging the characteristics; this is true even in the direction of least transparency.

Now suppose

$$\Sigma = a\Delta_i \quad \text{for} \quad \mu_{i+1} < |\mu| < \mu_i \tag{11}$$

The step function (11) should be regarded as a natural approximation of all possible dependences $\Sigma = \Sigma(\mu)$.

Integrating, we find a solution of the form (6) with

$$J = -\frac{\omega_0}{ab}\left\{1 - b\sum_i \Delta_i\left(\frac{\mu_i}{\Delta_i^2 - \mu_i^2\omega_0^2/a^2} - \frac{\mu_{i+1}}{\Delta_i^2 - \mu_{i+1}^2\omega_0^2/a^2}\right)\right\}^{-1} e^{-\omega_0|x|} \tag{12}$$

At the same time

$$\omega_0^2 \approx 3\,\frac{1-b}{b}\,a^2\left[\sum_i \Delta_i^{-2}(\mu_i^3 - \mu_{i+1}^3)\right]^{-1} \tag{13}$$

The expression (13) holds if the absorption is weak. It is clear that a gap of arbitrarily small but finite width with a vanishing value of Δ results in a reduction of ω_0 to zero. This means that the particle density does not decrease in the asymptotic region of large $|x|$. It can also be seen from (13) that deep local dips in the cross sections reduce ω_0 significantly only when Δ_i tends to zero faster than the width, $\mu_i - \mu_{i+1}$, of the corresponding interval of angles. It is therefore clear that in the case described by (9) we have $\omega_0 \to 0$ as $\Delta \to 0$, whereas if the cross sections satisfy the law

$$\Sigma = a\begin{cases}\Delta^k, & |\mu| < \Delta\\ \mu^k, & |\mu| > \Delta\end{cases} \quad (k < 1) \tag{14}$$

this is not the case.

Note that all the above formulas, and also those used later, go over into the formulas for an isotropic system (see, for example, [2]) in the corresponding special cases.

We now solve the problem for an anisotropic scattering law. If the distribution function is independent of the azimuth, the original single-velocity transport equation can be written in the form

$$\mu\frac{\partial f(x,\mu)}{\partial x} + \Sigma(\mu)f(x,\mu) - \sum_{m=0}^{\infty}\frac{2m+1}{2}g_m P_m(\mu)\int_{-1}^{1}P_m(\mu')\Sigma_s(\mu')f(x,\mu')\,d\mu' = \frac{1}{2}\delta(x)R(\mu) \tag{15}$$

The function $R(\mu)$ determines the distribution of the created particles over the directions. The scattering indicatrix is nonspherical; g_m are the coefficients of the expansion in Legendre polynomials of the angular scattering function. It is to be assumed that

$$g_0 = 1, \quad \int_{-1}^{1} R(\mu)\, d\mu = 1$$

Under these conditions, Eq. (15) is satisfied by

$$f(x, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ R(\mu) + \sum_{m=0}^{\infty} \frac{2m+1}{2} g_m h_m(\omega) P_m(\mu) \right\} [\Sigma(\mu) + i\omega\mu]^{-1} e^{i\omega x} d\omega \tag{16}$$

where

$$h_m(\omega) = \int_{-1}^{1} \left\{ R(\mu) + \sum_{n=0}^{\infty} \frac{2n+1}{2} g_n h_n(\omega) P_n(\mu) \right\} [\Sigma + i\omega\mu]^{-1} \Sigma_s(\mu) P_m(\mu)\, d\mu \tag{17}$$

Equation (17) gives a system of algebraic equations for finding $h_m(\omega)$; the order of the system is determined by the highest index for which $g_m \neq 0$. If

$$g_1 = g, \quad g_{m>1} = 0$$

then (16) becomes

$$f(x, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(R + \frac{1}{2} h_0 + \frac{3}{2} g h_1 \mu \right) (\Sigma + i\omega\mu)^{-1} e^{i\omega x} d\omega \tag{18}$$

At the same time

$$h_0 = D_0 / D, \qquad h_1 = D_1 / D \tag{19}$$

where

$$D = \left(1 - \frac{1}{2} \int_{-1}^{1} V d\mu\right)\left(1 - \frac{3}{2} g \int_{-1}^{1} V \mu^2 d\mu\right) - \frac{3}{4} g \left(\int_{-1}^{1} V \mu d\mu\right)^2$$

$$D_0 = \left(1 - \frac{3}{2} g \int_{-1}^{1} V \mu^2 d\mu\right) \int_{-1}^{1} V R d\mu + \frac{3}{2} g \int_{-1}^{1} V \mu d\mu \int_{-1}^{1} V R \mu d\mu$$

$$D_1 = \left(1 - \frac{1}{2} \int_{-1}^{1} V d\mu\right) \int_{-1}^{1} V R \mu d\mu + \frac{1}{2} \int_{-1}^{1} V \mu d\mu \int_{-1}^{1} V R d\mu$$

$$V(\mu) = \Sigma_s(\mu) [\Sigma(\mu) + i\omega\mu]^{-1} \tag{20}$$

Substituting (19) and (20) into (18), we obtain the spatial and angle distribution of the density of neutrons for arbitrary dependences $R(\mu)$, $\Sigma(\mu)$, $\Sigma_s(\mu)$. Integrating (18) with respect to μ, we find the spatial variation of $\varphi(x)$, the global particle density. To integrate with respect to ω, we must find the poles of the function

$$\frac{e^{i\omega x}}{D} \int_{-1}^{1} (DR + {}^1/_2 D_0 + {}^3/_2 D_1 g \mu)(\Sigma + i\omega\mu)^{-1} d\mu \tag{21}$$

The most interesting poles of (21) are those that are determined by the equation

$$D = 0;$$

the reason is this: those for which $|\omega_p|$ is smallest are responsible for the asymptotic behavior of the global density of neutrons in the case of weak absorption ($\Sigma - \Sigma_s \ll \Sigma$) under very general assumptions concerning the form of the functions $R(\mu)$, $\Sigma(\mu)$, and $\Sigma_s(\mu)$. Assuming that $|\omega_p|$ is small compared with Σ

(weak absorption), making elementary transformations in the expression (20) for D, and recalling that the functions $\Sigma(\mu)$ and $\Sigma_s(\mu)$ are even, we find

$$\omega_p^2 = -A/B$$

$$A = \left(1 - \int_0^1 \frac{\Sigma_s}{\Sigma} d\mu\right)\left(1 - 3g\int_0^1 \frac{\Sigma_s\mu^2}{\Sigma} d\mu\right)$$

$$B = \left(1 - 3g\int_0^1 \frac{\Sigma_s\mu^2}{\Sigma} d\mu\right)\int_0^1 \frac{\Sigma_s\mu^2}{\Sigma^3} d\mu + 3g\left(1 - \int_0^1 \frac{\Sigma_s}{\Sigma} d\mu\right)\int_0^1 \frac{\Sigma_s\mu^4}{\Sigma^3} d\mu + 3g\left(\int_0^1 \frac{\Sigma_s\mu^2}{\Sigma^2} d\mu\right)^2$$

Thus, the asymptotic solution gives an exponential decrease with linear parameter, $|\omega_p|^{-1}$, that depends on $\Sigma_s(\mu)$, $\Sigma(\mu)$, and the degree of anisotropy of the scattering function in the laboratory frame. It is readily seen that an anisotropy of the scattering reduces $|\omega_p|$, i.e., it retards the decrease of $\varphi(x)$ with increasing $|x|$ for any dependences $\Sigma(\mu)$ and $\Sigma_s(\mu)$.

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